AMS256 Homework 2

1. Consider the model

$$y_{i,j} = \mu + \alpha_i + \beta_i + \epsilon_{i,j} \text{ for } i = 1, \ldots, a, \text{ and } j = 1, \ldots, b,$$

   (a) Write $\boldsymbol{X}$. What is the rank of $\boldsymbol{X}$? What is the dimension of $\mathcal{N}(\boldsymbol{X})$?

   (b) Find $\boldsymbol{X}^T\boldsymbol{X}$. Show that

$$\boldsymbol{G} = \begin{bmatrix} 1/(ab) & 0 & 0 \\ -1/(ab)\mathbf{1}_a & 1/b\boldsymbol{I}_a & \mathbf{0} \\ -1/(ab)\mathbf{1}_b & \mathbf{0} & 1/a\boldsymbol{I}_b \end{bmatrix}$$

   is a generalized inverse of $\boldsymbol{X}^T\boldsymbol{X}$.

   (c) Assume that $a = 3$, $b = 4$. Show that $\boldsymbol{u}_1 = (1, -1, -1, -1, 0, 0, 0, 0)^T$ and $\boldsymbol{u}_2 = (1, 0, 0, 0, -1, -1, -1, -1)^T$ form a basis for $\mathcal{N}(\boldsymbol{X})$.

2. (Monahan) To evaluate a new curriculum in biology, two teachers each taught two classes using the old curriculum and three teachers taught two classes with the new. The responses, y_{ijk} is the average score for the class on the final. The data are:

			n_{ij}	y_{ij1}	y_{ij2}
$i = 1$(old)	$j = 1$	Dr. Able	2	100	80
	$j = 2$	Dr. Baker	2	80	80
$i = 2$(new)	$j = 1$	Dr.Able	2	110	90
	$j = 2$	Dr. Brown	2	100	140
	$j = 3$	Dr. Charles	2	110	150

Consider a nested model;

$$y_{ijk} = \mu + \alpha_i + \beta_{ij} + \epsilon_{ijk},$$

with $\mathrm{E}(\epsilon_{ijk}) = 0$.

   (a) Write this as a linear model of the form $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$. What is $r = rank(\boldsymbol{X})$?

   (b) Write the normal equations and find all solutions.

   (c) Give a set of basis vectors for $\mathcal{N}(\boldsymbol{X})$.

   (d) Give a list of r linearly independent estimable functions, $\boldsymbol{\lambda}^T\boldsymbol{\beta}$ and give the LSE for each one.

   (e) Show that $\alpha_1 - \alpha_2$ is not estimable.

   (f) For which of the following sets of parameter values $\boldsymbol{\beta}$ is the mean vector, $\boldsymbol{X\beta}$ the same?

$$\begin{aligned} \boldsymbol{\beta}_1 &= (100, 0, 0, 0, 0, 0, 0, 0)^T \\ \boldsymbol{\beta}_2 &= (90, 0, 10, 10, 0, 10, 20, 20)^T \\ \boldsymbol{\beta}_3 &= (50, 40, 30, 30, 10, 20, 20, 20)^T \\ \boldsymbol{\beta}_4 &= (80, 20, 10, 10, 0, 10, 20, 20)^T \\ \boldsymbol{\beta}_5 &= (90, 0, 20, 10, 0, 0, 10, 10)^T \end{aligned}$$

(g) For the parameter vectors in (f) which give the same $\boldsymbol{X\beta}$, show that the estimable functions you gave in (e) have values of $\boldsymbol{\lambda}^T\boldsymbol{\beta}$ that are the same.

3. Consider the regression model,

$$\mathrm{E}(Y_i) = \beta_0 + \beta_1 x_i + \beta_2(3x_i^2 - 2), \quad i = 1, 2, 3,$$

where $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$. Find the LSEs of β_0, β_1 and β_2. Find the LSEs of β_0 and β_1 assuming that $\beta_2 = 0$.